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EXTREME VALUE LIMIT LAWS IN THE NONIDENTICALLY DISTRIBUTED CASE

BY

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Dedicated to the Memory of N. V. Smirnov (1900–1966)

ABSTRACT

Let $X_1, \ldots, X_n$ be independent random variables, let F_i be the distribution function of X_i $(1 \leqq i \leqq n)$ and let $X_{1n} \leqq \cdots \leqq X_{nn}$ be the corresponding order statistics. We consider the statistics X_{kn}, where $k = k(n)$, $k/n \to 1$ and $n - k \to \infty$. Under some additional restrictions concerning the behaviour of the sequences $\{a_n > 0,\ b_n,\ k(n),\ F_n\}$ we characterize the class of all distribution functions H such that

$$\mathrm{Prob}\{(X_{kn} - b_n)/a_n < x)\} \to H.$$

1. Introduction

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables and let $X_{1n} \leqq \cdots \leqq X_{nn}$ be the corresponding order statistics. For every k $(1 \leqq k \leqq n)$ let F_{kn} denote the distribution function (df) of X_{kn}.

Much work has been done on the asymptotic behaviour of the statistics X_{kn}, when normalized as $n \to \infty$. The study of the maximal term X_{nn} was started by M. Fréchet [3], and definitive results were obtained by B. V. Gnedenko [5]. The class of the well-known extreme value limit laws of Gnedenko will be denoted here by A.

N. V. Smirnov [16] considered the term X_{kn}, where $k = k(n)$ is a function of n. He investigated the possible proper limits of a sequence of the form $F_{kn}(a_n x + b_n)$, under the condition that

$$k/n \to \lambda \tag{1.1}$$

and $a_n > 0$, b_n are numerical sequences. A sequence of terms X_{kn} was called by Smirnov a sequence of *central* terms, if $0 < \lambda < 1$ and *extreme* terms, if $\lambda = 0$ or

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$\lambda = 1$. Under some additional assumption on the sequence $k(n)$, Smirnov found the class of all possible limits for the central terms, but his investigation of the extreme terms was restricted to the cases when $k = \text{const}$ (the left extreme terms with $\lambda = 0$) or, what is essentially the same, when $p = n - k = \text{const}$ (the right extreme terms with $\lambda = 1$). Denoting this latter class by A_p, we can formulate Smirnov's result as follows:

The df H belongs to A_p if and only if it has the form

$$H(x) = A(x) \sum_{s=0}^{p-1} (-\log A(x))^s / s!, \tag{1.2}$$

where $A(x)$ is any df of Gnedenko's class A.

The study of the limits for the extreme terms in the remaining case, when

$$k \to \infty \quad \text{and} \quad p = n - k \to \infty, \tag{1.3}$$

was initiated by D. M. Chibisov [2]. Under some rather strong restrictions on the sequence $k(n)$ Chibisov proved that a df H can be a limit in the case (1.3) if and only if it has the form

$$H(x) = \Phi(-\log|\log A(x)|),$$

where Φ is the standard normal df and $A(x)$ is any df of the class A.

Chibisov's class was obtained in [13] under less restrictive conditions. But the best results in this direction belong to A. A. Balkema and L. de Haan [1]. They developed a general theory of limit laws for order statistics in case (1.3). In particular they obtained Chibisov's class under the following condition concerning the $k(n)$:

$$k(n+1) - k(n) = o(\min\{\sqrt{k(n)}, \sqrt{n - k(n)}\}).$$

Moreover, they proved that if, apart from (1.1), no other restriction is imposed on $k(n)$, then for a given λ $(0 \leqq \lambda \leqq 1)$ any df may be a limit law for some X_{kn}.

It seems to be of interest (cf. [8], p. 183) to study the asymptotic behaviour of the extreme order statistics in the more general case, when the underlying independent random variables $X_1, \dots, X_n$ are not necessarily identically distributed.

Let F_i be the df of X_i and F_{kn} — as above — the df of the statistics X_{kn}. For every df H we introduce the notation

$$\underline{H} = \inf\{x : H(x) > 0\}, \quad \bar{H} = \sup\{x : H(x) < 1\}. \tag{1.4}$$

A non-trivial extension of the class A is the class G of all df's H such that

(1.5) $$F_{nn}(a_n x + b_n) \to H(x),$$

under the additional condition that for every $x > \underline{H}$

(1.6) $$\min_{1 \leq i \leq n} F_i(a_n x + b_n) \to 1.$$

The class G was studied in [7] and [9–12]. We proved that the df H belongs to G if and only if $\log H$ is a concave function or if $\bar{H} < \infty$ and the function $\log H(\bar{H} - e^{-x})$ is concave.

Let us denote by G_p a similar extension of Smirnov's class A_p. It was proved in [15] that H belongs to G_p if and only if it can be presented by Smirnov's formula (1.2), where $A(x)$ is replaced by $G(x)$, i.e., any df of the class G.

A comprehensive view of the evolution of the subject is given by J. Galambos in his monograph [4].

It is the purpose of the present paper to consider the right extreme terms in case (1.3), where

(1.7) $$p/n \to 0$$

and the underlying independent random variables $X_1, \ldots, X_n$ are not necessarily identically distributed.

In order to achieve an exact description of a class of possible limit df's in our more general situation, we are compelled to impose additional restrictions besides the condition of Balkema–de Haan. Denote

(1.8) $$u_n = \left[\sum_{i=1}^{n} F_i - n\right] \Big/ \sqrt{p(n)} + \sqrt{p(n)}, \quad f_n = \frac{1}{n} \sum_{i=1}^{n} F_i$$

and let us introduce the following

DEFINITION 1.1. Let S be the class of all proper df's H (i.e. $\underline{H} < \bar{H}$), which have the following property:

There exists a sequence of independent random variables X_n with corresponding df's F_n, numerical sequences $a_n > 0$, b_n and positive integers $k(n)$ such that in addition to (1.3) and (1.5)–(1.7), the following conditions are satisfied:

(a) The condition of Balkema–de Haan, which in our notation can be written in the form

(1.9) $$\sqrt{p(n+1)} - \sqrt{p(n)} \to 0.$$

(b) For every positive integer n and r and every $x < \bar{f}_n$

(1.10) $$u_n \geqq u_{n+r},$$

where $\bar{f}_n$ is defined by (1.8) and (1.4).

(c) The numerical sequences $\{a_n, b_n, p(n)\}$ are related in the following way: if for some integer-valued function $m = m(n)$

(1.11) $$a_{n+m}/a_n \to \alpha \quad \text{or} \quad (b_{n+m} - b_n)/a_n \to \beta,$$

where $0 < \alpha < \infty$, $0 \leqq \beta < \infty$, then also

(1.12) $$p(n+m)/p(n) \to 1.$$

We prove the following

THEOREM 1.1. *The df H belongs to S if and only if it has the form*

(1.13) $$H = \Phi(u),$$

where Φ is the standard normal df and either

$$\bar{H} = \infty \text{ and } u \text{ is a concave function}$$

or

$$\bar{H} < \infty \text{ and the function } u(\bar{H} - e^{-x}) \text{ is concave.}$$

We will start with the particular cases $a_n \equiv 1$ (Section 3) and $b_n \equiv 0$ (Section 4), since these cases seem to be of interest by themselves, and since we intend to prove that the class S (when the a_n and b_n are varying simultaneously) is the union of the classes, which correspond to the above particular cases.

Finally, let us point out that here and in the sequel convergence of sequences of monotone functions means weak convergence, i.e., convergence at each continuity point of the limit function.

2. Preliminary remarks

2.1. In our case, when the underlying independent random variables are not identically distributed, the df F_{kn} of the statistics X_{kn} has quite a complicated form:

$$F_{kn} = \left\{1 + \sum_{s=1}^{n-k} \sum^* \prod_{i \in (n,s)} (1/F_i - 1)\right\} \prod_{i=1}^{n} F_i,$$

where (n, s) is any set of s integers in $\{1, \dots, n\}$ and the summation Σ^* is over all such sets. We proved [14] a proposition, which may simplify the investigation of X_{kn}.

THEOREM 2.1 (cf. [16], Theorem 4). *Let H be a proper df, and let* (1.3) *hold. Assume for every x* ($\underline{H} < x < \bar{H}$)

$$F_i(a_n x + b_n) \to \lambda = \text{const}$$

uniformly in i ($1 \leqq i \leqq n$). *Put*

$$u_n = \sqrt{n}\left(\sum_{i=1}^{n} F_i - k\right) \Big/ [k(n-k)]^{1/2}. \tag{2.1}$$

Then relation (1.5) *holds if and only if*

$$u_n(a_n x + b_n) \to u(x), \tag{2.2}$$

where the non-decreasing function u is determined from H by the equation (1.13).

If $\lambda = 1$, *then the expression* (2.1) *can be rewritten in the form* (1.8).

2.2. By (1.13) the function u in (2.2) is a non-decreasing function, which assumes finite or infinite values in $(-\infty, \infty)$ and satisfies the conditions $u(-\infty) = -\infty$, $u(+\infty) = +\infty$. Such a function will be called here an *s-function* (sf).

For every sf we denote

$$\underline{u} = \inf\{x : u(x) > -\infty\}, \quad \bar{u} = \sup\{x : u(x) < +\infty\}. \tag{2.3}$$

An sf will be called proper if $\underline{u} < \bar{u}$. It follows from (1.4), (1.13) and (2.3) that H is a proper df if and only if the corresponding sf u is proper, since $\underline{u} = \underline{H}$, $\bar{u} = \bar{H}$.

Using Theorem 2.1, we will investigate in the sequel the possible proper limits in (2.2).

2.3. The notion of type of sf's can be introduced in the same way as was done in the domain of df's. The theorems of Khintchine and Gnedenko [6, §10, Theorems 1–2] about sequences of df's which converge to a proper type, can be extended to sequences of non-decreasing functions (not necessarily sf's), which converge to a proper sf. It follows from (1.8) that

$$u_{n+1}(x) = A_n u_n(x) + B_n(x),$$

where by (1.9) $A_n \to 1$ and $B_n(x) \to 0$. Therefore, if relations (1.3), (1.9) and (2.2) hold, then

$$u_{n+1}(a_n x + b_n) \to u(x).$$

Hence, by the theorems mentioned above, we conclude that if $u(x)$ is a proper sf, then

$$a_{n+1}/a_n \to 1, \quad (b_{n+1} - b_n)/a_n \to 0. \tag{2.4}$$

2.4. In contrast to the df F_{kn} the expression (1.8) makes sense for $p(n)$, which are not necessarily integers. The limits in (2.2) do not change if $p(n)$ is replaced by any function $g(n)$ such that $\sqrt{p(n)} - \sqrt{g(n)} \to 0$. Obviously, each such $g(n)$ will satisfy together with $p(n)$ the conditions (1.3), (1.7), (1.9) and (1.12).

2.5. Let c be any number. Put

$$\tilde{p}_n(n) = p(n) + c\sqrt{p(n)}, \quad \tilde{u}_n = \left[\sum_{i=1}^{n} F_i - n\right] \Big/ \sqrt{\tilde{p}(n)} + \sqrt{\tilde{p}(n)}.$$

If for some $\{a_n, b_n\}$ we have (2.2), then

$$\tilde{u}_n(a_n x + b_n) \to u(x) + 2c.$$

2.6. It follows from (1.8) that (1.10) holds if $u_n(x) < \sqrt{p(n)}$. Thus if relation (2.2) holds, then for any $x < \bar{u}$ we have from some n on $a_n x + b_n < \bar{f}_n$ and therefore, by assumption (1.10), for any positive integer r

$$u_{n+r}(a_n x + b_n) \leqq u_n(a_n x + b_n).$$

2.7. If for some x

$$a_{n+1}x + b_{n+1} \geqq a_n x + b_n, \quad F_n(a_n x + b_n) \to 1$$

and for each i fixed $F_i(a_n x + b_n) \to 1$, then also (1.6) holds.

3. The particular case $a_n \equiv 1$

Definition 3.1. Let us put $a_n \equiv 1$ in the conditions, which define the class S. Then we obtain a subset of S that we denote by S^*.

The class S^* was studied in [14], where we proved

Theorem 3.1. *The df H belongs to S^* if and only if it has the form* (1.13), *where u is a proper sf, which has the following property:*
for every $\beta > 0$ the difference

$$\varphi_\beta(x) = u(x) - u(x + \beta) \tag{3.1}$$

is a non-decreasing function in $(\underline{u}, \infty)$. In other words, u is a concave function.

For the sake of completeness, and particularly since our further considerations are closely connected to this theorem, we give here a sketch of the proof of the sufficiency.

By Theorem 2.1 it is enough to show that under assumption (3.1) there exists a triple $\{F_n, b_n, p(n)\}$, which satisfies all the conditions that appear in Definition 1.1, where $a_n \equiv 1$ in (1.6) and condition (1.5) is replaced by

$$u_n(x+b_n) \to u(x), \tag{3.2}$$

where u_n is given by (1.8).

It follows from (3.1) that $\bar{u}=\infty$. Indeed, assuming $\bar{u}<\infty$, let $\underline{u}<\xi<\xi+\beta<\eta<\bar{u}<\bar{u}+\beta$. Then we obtain $\varphi_\beta(\xi)>\varphi_\beta(\eta)=-\infty$, which is impossible, since $\varphi_\beta(x)$ is non-decreasing. Thus u increases continuously in $(\underline{u},\infty)$ and has there the one-sided derivatives $u^+(x)=u'(x+0)$ and $u^-(x)=u'(x-0)$. These properties of u enable us to construct effectively the required triple $\{F_n, b_n, p(n)\}$.

By (1.6) and (1.8), a finite number of terms of this triple may be chosen arbitrarily. Therefore, some relations which hold from some n on, can be treated here as valid for every n.

We introduce auxiliary numerical sequences c_n and d_n by

$$c_n = -n/\sqrt{p(n)}+\sqrt{p(n)}, \quad u(d_n)=\sqrt{p(n)}, \tag{3.3}$$

and define

$$F_n = 1-p(n)+p(n-1)+\sqrt{p(n)}u_n(x)-\sqrt{p(n-1)}u_{n-1}(x) \quad (p(0)=0), \tag{3.4}$$

where the u_n are non-decreasing functions, defined below depending on the behaviour of u on the right of $\underline{u}$. However, we always put

$$u_n(x)=c_n \quad \text{if } x<0, \qquad u_n(x)=\sqrt{p(n)} \quad \text{if } x>b_n+d_n, \tag{3.5}$$

that implies

$$F_n(x)=0 \quad \text{if } x<0, \qquad F_n(x)=1 \quad \text{if } x>b_n+d_n. \tag{3.6}$$

By (3.3)

$$\sum_{i=1}^{n} F_i = n-p(n)+\sqrt{p(n)}\,u_n.$$

Hence, $u_n(x)$ has the form (1.8) and it remains to define u_n in the interval $[0, b_n+d_n]$, and the sequences b_n and $p(n)$.

We consider four cases.

Case I: $\underline{u} = -\infty$. We define

$$u_n(x) = u(x - b_n) \quad \text{if } 0 \leqq x \leqq b_n + d_n,$$

(3.7) $$u(-b_n) = -\log n, \quad p(n) = b_n.$$

If $\underline{u} > -\infty$ we define

(3.8) $$p(n) = \sqrt{n}, \quad b_n = \sqrt[8]{n}.$$

We may assume $\underline{u} = 0$. Moreover, if in addition $u(0+) > -\infty$, then by Remark 2.5 we can assume $u(0+) = 0$.

Case II: $\underline{u} = 0$, $u(0+) = 0$, $u'(0+) < \infty$. We define the functions u_n by

$$u_n(x) = \begin{cases} b_n(x - b_n) & \text{if } 0 \leqq x < b_n, \\ u(x - b_n) & \text{if } b_n \leqq x \leqq b_n + d_n. \end{cases}$$

Case III: $\underline{u} = 0$, $u(0+) = 0$, $u'(0+) = \infty$. Let $g(x)$ be a continuous and non-increasing function in $(0, \infty)$, satisfying $g(x) \geqq u'(x)$. Let

(3.9) $$f(x) = g(x)/u(x),$$

(3.10) $$f(\beta_n) = \sqrt[8]{n},$$

then $0 < \beta_{n-1} - \beta_n < b_n - b_{n-1}$, $\beta_n \to 0$. We define

$$u_n(x) = \begin{cases} u(\beta_n) + g(\beta_n)(x - \beta_n - b_n) & \text{if } 0 \leqq x < b_n + \beta_n, \\ u(x - b_n) & \text{if } b_n + \beta_n \leqq x \leqq b_n + d_n. \end{cases}$$

Case IV: $\underline{u} = 0$, $u(0+) = -\infty$. We take a function $g(x)$ as in the previous case and we define the sequence β_n by (3.10), where now, instead of (3.9), the function $f(x)$ is given by

$$f(x) = -u(x)g(x).$$

The function u_n is defined as in the previous case.

We proved in [14] that by the above determination of the u_n the functions F_n, given by (3.4), are df's and the conditions (1.6) and (3.2) are satisfied.

It should be noted that the restriction (1.10), which appears in Definition 1.1 of the class S, is not needed for the proof of Theorem 3.1, and we introduced this restriction keeping in mind the proof of Theorem 1.1. Therefore, it remains to show that our u_n satisfy condition (1.10) too. Indeed, in each of the four considered cases we have $\bar{f}_n = b_n + d_n$. Denote $\Delta_n(x) = u_n(x) - u_{n+1}(x)$. Since $\bar{f}_n \leqq \bar{f}_{n+1}$, in order to prove (1.10) it is enough to show that

$$\Delta_n(x) \geqq 0 \quad \text{if } x < b_n + d_n. \tag{3.11}$$

However, the validity of (3.11) can be verified, taking into consideration the monotonicity of the numerical sequences, which appear in the definition of the $u_n(x)$.

For the next section we will need the following

THEOREM 3.2. *Let $u(x)$ be a non-decreasing and concave function which assumes finite or infinite values in $(-\infty, \infty)$ and satisfies the conditions $u(-\infty) = -\infty$, $u(+\infty) = l < \infty$. There exists a triple $\{F_n, b_n, p(n)\}$ which satisfies all the conditions that appear in the definition of the class S^*.*

PROOF. Without loss of generality we can assume (Remark 2.5) that $l > 0$. Denote

$$\underline{u} = \inf\{x : u(x) > -\infty\}, \quad \bar{u}_l = \sup\{x : u(x) < l\}.$$

We define the numerical sequences $\{b_n, p(n)\}$ by (3.7) if $\underline{u} = -\infty$ and by (3.8) if $\underline{u} > -\infty$. The df's F_n are defined by (3.4), where $u_n(x)$ are non-decreasing and continuous for $x \neq 0$. The $u_n(x)$ will be specified later. However, we will always have (3.5) where the sequence $\{c_n\}$ is given by (3.3) and the $\{d_n\}$ will be also defined later. Condition (3.5) implies (3.6) and it remains to define the $u_n(x)$ in the interval $[0, b_n + d_n]$ only. In order to define the u_n, we have to consider the various ways the $u(x)$ can behave on the left of $\bar{u}_l$ under the various assumptions concerning the behaviour of $u(x)$ on the right of $\underline{u}$, i.e., the cases I–IV, which were considered in the proof of Theorem 3.1.

We will consider a few cases under assumption I, i.e., when $\underline{u} = -\infty$. Fortunately, the passage from I to II–IV is very simple.

Let us consider in detail the case

(A; I): $\bar{u}_l = \infty$; $\underline{u} = -\infty$. Since u is concave, there exists by (3.7) a constant $c > 0$ such that

$$b_n = c \log n, \quad 0 < b_{n+1} - b_n < c \log \frac{n+1}{n} < c/n. \tag{3.12}$$

Denote

$$\gamma_n = \inf\{x : u'(x) \geqq 1/\sqrt{p(n)}\} = \sup\{x : u'(x) \leqq 1/\sqrt{p(n)}\}.$$

In our case $u'(x) \to 0$, when $x \to \infty$. Therefore

$$\gamma_n \geqq \gamma_{n-1} \to \infty \tag{3.13}$$

and

$$u^-(\gamma_n) = u'(\gamma_n - 0) \geqq 1/\sqrt{p(n)}, \quad u^+(\gamma_n) = u'(\gamma_n + 0) \leqq 1/\sqrt{p(n)}. \tag{3.14}$$

Denote

$$d_n = \gamma_n + p(n) - u(\gamma_n)\sqrt{p(n)}.$$

We can assume

$$d_n > d_{n-1} \to \infty, \tag{3.15}$$

since $u(\gamma_n)\sqrt{p(n)} - u(\gamma_{n-1})\sqrt{p(n-1)} < l(\sqrt{p(n)} - \sqrt{p(n-1)})$.

Finally, we define the $u_n(x)$ by

$$u_n(x) = \begin{cases} u(x - b_n) & \text{if } 0 \leqq x < b_n + \gamma_n, \\ (x - b_n - \gamma_n)/\sqrt{p(n)} + u(\gamma_n) & \text{if } b_n + \gamma_n \leqq x \leqq b_n + d_n. \end{cases} \tag{3.16}$$

Besides condition (3.13), the difference $(\gamma_n - \gamma_{n-1})$ is unrestricted. Therefore, taking into account (3.7), (3.13) and (3.15), the "critical points" of the expression (3.4) of F_n can be ordered in one of the two following ways:

$$-\infty < 0 < b_{n-1} + \gamma_{n-1} < b_n + \gamma_n < b_{n-1} + d_{n-1} < b_n + d_n < \infty \tag{3.17.a}$$

or

$$-\infty < 0 < b_{n-1} + \gamma_{n-1} < b_{n-1} + d_{n-1} < b_n + \gamma_n < b_n + d_n < \infty. \tag{3.17.b}$$

Let us assume first (3.17.a) and let us examine the F_n in the corresponding intervals.

If $0 \leqq x \leqq b_n + \gamma_n$, then F_n can be presented in the form

$$F_n = 1 - p(n) + p(n-1) + \sqrt{p(n)}[u(x - b_n) - u(x - b_{n-1})]$$
$$+ [\sqrt{p(n)} - \sqrt{p(n-1)}]u(x - b_{n-1}).$$

Since $u(x)$ is concave, the difference $u(x - b_n) - u(x - b_{n-1})$, and consequently $F_n(x)$, increases in the interval under consideration.

If $b_{n-1} + \gamma_{n-1} < x \leqq b_n + \gamma_n$, then

$$F_n = 1 - p(n) + p(n-1) + \sqrt{p(n)}\,u(x - b_n)$$
$$- [x - b_{n-1} - u(\gamma_{n-1}) + u(\gamma_{n-1})\sqrt{p(n-1)}].$$

Hence by (3.14)

$$F_n'(x-0) = \sqrt{p(n)}\,u^-(x - b_n) - 1 \geqq \sqrt{p(n)}\,u'(\gamma_n - 0) - 1 \geqq 0$$

and F_n does not decrease. The monotonicity of F_n in the other intervals is obvious. The F_n are continuous for $x \neq 0$ and we conclude that F_n does not decrease in $(0, \infty)$.

It follows from (3.12) and (3.16) that

$$F_n(0+) \to 1. \tag{3.18}$$

Thus we can assume that $F_n(x) > 0$ for $x > 0$ and by (3.6) we obtain that F_n are df's.

By Remark 2.7, relation (3.18) also implies the validity of (1.6). Now, for any x we have from some n on $0 < x + b_n < \gamma_n + b_n$ and by (3.16) we have from some n

$$u_n(x + b_n) = u(x). \tag{3.19}$$

It remains to prove that our u_n satisfy condition (3.11) too. Obviously, $\Delta_n(x) > 0$ if $x \leqq b_n + \gamma_n$. If $b_n + \gamma_n < x \leqq b_{n+1} + \gamma_{n+1}$, then

$$\Delta_n(x) = (x - b_n - \gamma_n)/\sqrt{p(n)} + u(\gamma_n) - u(x - b_n).$$

It is easy to observe that

$$\Delta_n(x) > 0 \quad \text{if } b_n + \gamma_n \leqq x \leqq b_{n+1} + \gamma_n.$$

On the other hand, in the interval $[b_{n+1} + \gamma_n, b_{n+1} + \gamma_{n+1}]$ we have by (3.14)

$$\Delta_n'(x+0) = 1/\sqrt{p(n)} - u^+(x - b_{n+1}) \geqq 1/\sqrt{p(n)} - u'(\gamma_n + 0) \geqq 0.$$

Hence we conclude that $\Delta_n(x) \geqq 0$ if $x \leqq b_{n+1} + \gamma_{n+1}$.

Finally, since $\Delta_n'(x) > 0$ for $b_{n+1} + \gamma_{n+1} < x < b_n + d_n$, we obtain the validity of (3.11).

If we assume (3.17.b) we argue in a similar way.

If $\bar{u}_l < \infty$, we may assume $\bar{u}_l = \gamma > 0$.

(B; I): $0 < \bar{u}_l = \gamma < \infty$, $u'(\gamma - 0) = 0$; $\underline{u} = -\infty$. We define the numerical sequences γ_n, d_n and the functions u_n as in the previous case. However, now we will have

$$\gamma_{n-1} \leqq \gamma_n \to \gamma.$$

In the present case we have (3.17.a) and the argument is the same as in case (A; I). Now, if $x < \gamma$, then from some n on we have $0 < x + b_n < b_n + \gamma_n$ and by (3.16) we obtain (3.19). If $x > \gamma$, then from some n on $b_n + \gamma_n < x + b_n < b_n + d_n$ and by (3.16)

$$u_n(x + b_n) = (x - \gamma_n)/\sqrt{p(n)} + u(\gamma_n) \to u(\gamma) = l.$$

(C; I): $0 < \bar{u}_l = \gamma < \infty$, $u'(\gamma - 0) > 0$; $\underline{u} = -\infty$. We define

$$u_n(x) = \begin{cases} u(x - b_n) & \text{if } 0 \leqq x \leqq b_n + \gamma, \\ (x - b_n - \gamma)/\sqrt{p(n)} + l & \text{if } b_n + \gamma < x \leqq b_n + d_n, \end{cases}$$

where $d_n = \gamma + p(n) - l\sqrt{p(n)}$.

Let us point out that for $b_{n-1} + \gamma < x \leqq b_n + \gamma$ we have

$$F'_u(x - 0) = \sqrt{p(n)}\, u^-(x - b_n) - 1 \geqq \sqrt{p(n)}\, u'(\gamma - 0) - 1$$

and $F_n(x)$ increases from some n on, since $u'(\gamma - 0) > 0$.

If $\underline{u} > -\infty$, we can assume one of the cases II–IV and we obtain the definition of $u_n(x)$ by combining our present formulas with the determination of the $u_n(x)$ on the right of $\underline{u}$, as was done in the proof of Theorem 3.1.

Thus, in case (A; II) we define

$$u_n(x) = \begin{cases} b_n(x - b_n) & \text{if } 0 \leqq x \leqq b_n, \\ u(x - b_n) & \text{if } b_n < x \leqq b_n + \gamma_n, \\ (x - b_n - \gamma_n)/\sqrt{p(n)} + u(\gamma_n) & \text{if } b_n + \gamma_n < x \leqq b_n + d_n\, ; \end{cases}$$

and in the case (A; III) or (A; IV):

$$u_n(x) = \begin{cases} u(\beta_n) + g(\beta_n)(x - \beta_n - b_n) & \text{if } 0 \leqq x \leqq b_n + \beta_n, \\ u(x - b_n) & \text{if } b_n + \beta_n < x \leqq b_n + \gamma_n, \\ (x - b_n - \gamma_n)/\sqrt{p(n)} + u(\gamma_n) & \text{if } b_n + \gamma_n < x \leqq b_n + d_n. \end{cases}$$

In a similar way we extend the definition of the $u_n(x)$ in cases (B) and (C).

Finally, there remains the case

(D): $\bar{u}_l = \underline{u} = 0$. We have

$$u(x) = \begin{cases} -\infty & \text{if } x < 0, \\ l & \text{if } x > 0. \end{cases}$$

We define the $u_n(x)$ by

$$u_n(x) = \begin{cases} b_n(x - b_n) & \text{if } 0 \leqq x \leqq b_n, \\ (x - b_n)/\sqrt{p(n)} + l & \text{if } b_n < x \leqq b_n + d_n, \end{cases}$$

where $d_n = p(n)$.

This completes the proof.

4. The particular case $b_n \equiv 0$

LEMMA 4.1. *Let*

$$u_n(a_n x) \to u(x), \tag{4.1}$$

where u_n is given by (1.8) *and $u(x)$ is a proper sf. Let*

$$\min_{1 \leqq i \leqq n} F_i(a_n x) \to 1 \tag{4.2}$$

for every $x > \underline{u}$. Then either

$$a_n \to \infty \quad \textit{and} \quad \underline{u} \geqq 0$$

or

$$a_n \to 0 \quad \textit{and} \quad \bar{u} \leqq 0.$$

PROOF. Assume that for some subsequence $a_{n'} \to a$, where $0 \leqq a \leqq \infty$. If $0 < a < \infty$, then by the theorems of Khintchine and Gnedenko (Remark 2.3) we also have $u_{n'}(ax) \to u(x)$, and for every i and $x > \underline{u}$, $F_i(ax) = 1$. Thus by (4.1) we conclude that $u(x) = \infty$ for $x > \underline{u}$. Hence $\underline{u} = \bar{u}$ which is impossible, since u is a proper sf.

Let $a = \infty$. Assuming $\underline{u} < 0$, we shall have $F_i(a_{n'}x) \to 0$ for every i and $\underline{u} < x < 0$, which contradicts (4.2).

If $a = 0$, then by (4.2) we have $F_i(0+) = 1$ for every i and consequently $u_{n'}(a_{n'}x) = \sqrt{p(n')} \to \infty$ for $x > 0$, i.e. $\bar{u} \leqq 0$.

Obviously, the coexistence of both partial limits (0 and ∞) is excluded by the assumption $\underline{u} < \bar{u}$.

Now, taking into account Theorem 2.1, we obtain that if the df H belongs to S with $b_n \equiv 0$, then either

$$a_n \to \infty \quad \text{and} \quad \underline{H} \geqq 0 \tag{4.3}$$

or

$$a_n \to 0 \quad \text{and} \quad \bar{H} \leqq 0. \tag{4.4}$$

DEFINITION 4.1. Let us put $b_n \equiv 0$ in the conditions, which define the class S. Then we obtain a subclass of S, which is denoted here by S^+ if we have (4.3) and by S^- if we have (4.4).

The classes S^+ and S^- are not closed under translations. We denote by $\hat{S}^+$ and $\hat{S}^-$ the corrresponding extended classes, that contain all the types that are represented in S^+ and S^-, respectively.

4.1. *The classes S^+ and $\hat{S}^+$*

THEOREM 4.1. *The df H belongs to S^+ if and only if it has the form* (1.13), *where u is a proper sf, having the following properties: $\underline{u} \geqq 0$, and for every $\alpha > 1$ the difference*

$$\varphi_\alpha(x) = u(x) - u(\alpha x) \tag{4.5}$$

is a non-decreasing function in $(\underline{u}, \infty)$.

PROOF. *Sufficiency*. It follows from (4.5) that $\bar{u} = \infty$. Indeed, assuming $\bar{u} < \infty$, let

$$0 \leqq u \leqq \xi < \eta < \bar{u}, \qquad 1 < \bar{u}/\eta < \alpha < \bar{u}/\xi.$$

Then we obtain $\varphi_\alpha(\xi) > \varphi_\alpha(\eta) = -\infty$, which is impossible, since $\xi < \eta$. Denote

$$v(x) = u(e^x). \tag{4.6}$$

Obviously, $v(+\infty) = u(+\infty) = +\infty$. If $\underline{u} > 0$, then $v(x) = -\infty$ for $x < \log \underline{u}$. If $\underline{u} = 0$, then $v(-\infty) = u(0+)$. But if $\underline{u} = 0$, then necessarily $u(0+) = -\infty$. Indeed, the assumption $|u(0+)| < \infty$ implies $\varphi_\alpha(0+) = 0$ and consequently $\varphi_\alpha(x) = 0$ for every $x > 0$, since $\alpha x > x$ and $\varphi_\alpha(x)$ is non-decreasing. Hence $u(x) = \text{const}$ for $x > 0$, which is impossible, since u is an sf. Thus we have proved that $v(x)$ is a proper sf.

Now, let us substitute in (4.5) e^x instead of x and put $\alpha = e^\beta$. Then by (4.6) the difference $v(x) - v(x + \beta)$ does not decrease. Hence, by Theorem 3.1 the df $H = \Phi(v)$ belongs to S^* and there exists a triple $\{\tilde{F}_n, b_n, p(n)\}$ which satisfies all the conditions that appear in Definition 3.1. In particular we have

$$\tilde{u}_n(x + b_n) = \left[\sum_{i=1}^{n} \tilde{F}_i(x + b_n) - n\right] \Big/ \sqrt{p(n)} + \sqrt{p(n)} \to v(x), \tag{4.7}$$

$$\min_{1 \leqq i \leqq n} \tilde{F}_i(x + b_n) \to 1 \quad \text{for } x > \underline{v}, \tag{4.8}$$

where $b_n \to \infty$. Let us define

$$F_i(x) = \tilde{F}_i(\log x) \quad \text{for } x > 0, \qquad a_n = \exp(b_n).$$

Then putting $\log x$ instead of x in (4.7)–(4.8) we obtain (4.1)–(4.3) and therefore $H = \Phi(u)$ belongs to S^+.

Necessity. By Remark (2.3) we have (2.4). Since $a_n \to \infty$, there exists for every $\alpha > 1$ an integer-valued function $m = m_\alpha(n)$ such that

$$a_{n+m}/a_n \to \alpha. \tag{4.9}$$

The subsequence $u_{n+m}(a_{n+m}x)$ can be presented in the form

$$u_{n+m}(a_{n+m}x) = [p(n)/p(n+m)]^{1/2}u_n(a_{n+m}x) + \varphi_n(x), \tag{4.10}$$

where

$$\varphi_n(x) = \left[\sum_{i=n+1}^{n+m} F_i(a_{n+m}x) + p(n+m) - p(n) - m\right] \Big/ \sqrt{p(n+m)}$$

is a non-decreasing function. By (4.9), when $n \to \infty$ we have (1.12) and we obtain from (4.10) the inequality (4.5), where $\varphi_\alpha(x)$ is the limit of $\varphi_n(x)$.

Our argument implies

THEOREM 4.2. *H belongs to S^+ if and only if $H(e^x)$ belongs to S^*.*

COROLLARY 4.1. *The class $\hat{S}^+$ is a proper subset of S^*.*

PROOF. If $H = \Phi(u)$ belongs to S^+, then by Theorem 4.2, $\Phi(u(e^x))$ belongs to S^* and therefore $u(e^x)$ is a concave function. Hence, since e^x is convex, $u(x)$ must be concave and consequently $H = \Phi(u)$ belongs to S^*. Obviously, also $\hat{S}^+$ belongs to S^*, since S^* is closed under translations.

Although S^+ is not closed under translations, the following proposition is valid:

THEOREM 4.3. *If H belongs to S^+, then for any $b > 0$ also $H(x-b)$ belongs to S^+.*

PROOF. We have to show that if the sf u satisfies condition (4.5), then for any $\alpha > 1$ and $b > 0$ the difference $u(x-b) - u(\alpha x - b)$ is also a non-decreasing function. Indeed, let us assume that for some $b_1 > 0$, $\alpha_1 > 1$, $y_1 > x_1 > \underline{u} + b_1 > 0$ we have

$$u(y_1 - b_1) - u(\alpha_1 y_1 - b_1) < u(x_1 - b_1) - u(\alpha_1 x_1 - b_1). \tag{4.11}$$

By (4.5), for any $\alpha > 1$

$$u(y_1-b_1)-u(\alpha y_1-\alpha b_1)\geqq u(x_1-b_1)-u(\alpha x_1-\alpha b_1).$$

In particular, for $\alpha=(\alpha_1 y_1-b_1)/(y_1-b_1)>1$ we obtain from the last inequality that

$$u(y_1-b_1)-u(\alpha_1 y_1-b_1)\geqq u(x_1-b_1)-u[(x_1-b_1)(\alpha_1 y_1-b_1)/(y_1-b_1)]. \tag{4.12}$$

But it is easy to verify that

$$\alpha_1 x_1-b_1>(x_1-b_1)(\alpha_1 y_1-b_1)/(y_1-b_1).$$

Thus from (4.12) we obtain

$$u(y_1-b_1)-u(\alpha_1 y_1-b_1)\geqq u(x_1-b_1)-u(\alpha_1 x_1-b_1),$$

which contradicts assumption (4.11).

Finally, let us point out that $H\in S^+$ and $\underline{H}>0$ does not imply that also $H(\underline{H}+x)\in S^+$. Indeed, let us consider for example the df $H=\Phi(u)$, where

$$u(x)=\begin{cases}-\infty & \text{if } x<1,\\ \log x & \text{if } x>1;\end{cases}$$

obviously, $H\in S^+$, but $H(x+1)$ does not belong to S^+, since

$$[\log(e^x+1)]''>0.$$

This phenomenon can be explained by the fact that if $H\in S^+$ and $\underline{H}=0$, then as we have seen necessarily $H(0+)=0$. On the other hand, if $H\in S^+$, then for any h ($\underline{H}<h<\bar{H}$) the df

$$\tilde{H}(x)=\begin{cases}0 & \text{if } x<h,\\ H(x) & \text{if } x>h,\end{cases}$$

also belongs to S^+.

4.2. *The classes S^- and $\hat{S}^-$*

THEOREM 4.4. *The df H belongs to S^- if and only if it has the form* (1.13), *where u is a proper sf, having the following properties: $\bar{u}=0$, and for every $0<\alpha<1$ the difference* (4.5) *is a non-decreasing function in* $(\underline{u},0)$.

PROOF. *Sufficiency*. It follows from (4.5) that $\bar{u}=0$. Indeed, assuming $\bar{u}<0$, let

$$\underline{u}<\xi<\eta<-\sqrt{\bar{u}\xi},\qquad \eta/\xi<\alpha<\bar{u}/\eta<1.$$

Then we have $\underline{u} < \xi < \alpha\xi < \eta < \bar{u} < \alpha\eta < 0$ and consequently $\varphi_\alpha(\xi) > \varphi_\alpha(\eta) = -\infty$, which is impossible. Denote

$$v(x) = u(-e^{-x}).$$

If we replace x by $-e^{-x}$ in (4.5) and set $\alpha = e^{-\beta}$, we find that $v(x)$ is a concave function, since the difference $v(x) - v(x+\beta)$ does not decrease. We have $v(-\infty) = u(-\infty) = -\infty$ and $v(+\infty) = u(0-)$. Using Theorem 3.1 if $u(0-) = \infty$, and Theorem 3.2 if $(u0-) < \infty$, we can affirm the existence of a triple $\{F_n, b_n, p(n)\}$, which satisfies relations (4.7)–(4.8), where $b_n \to \infty$. Now we replace the variable x in (4.7)–(4.8) by $-\log(-x)$, we set $b_n = -\log a_n$ and define $F_i = \tilde{F}_i(-\log(-x))$. Then we obtain (4.1), (4.2) and (4.4), and we conclude that $H = \Phi(u)$ belongs to S^-.

Necessity. By Remark 2.3 we have (2.4). Since $a_n \to 0$, there exists for every $0 < \alpha < 1$ an integer-valued function $m = m_\alpha(n)$ such that we have (4.9). We obtain the necessity of (4.5) by the same argument as in the proof of Theorem 4.1.

We have seen above that if H belongs to S^-, then $\bar{H} = 0$. Hence, H belongs to $\hat{S}^-$ if and only if $\bar{H} < \infty$ and the df $H(\bar{H} + x)$ belongs to S^-. Thus, by the argument of the proof of Theorem 4.4 we conclude

THEOREM 4.5. *The df H belongs to $\hat{S}^-$ if and only if $\bar{H} < \infty$ and H has the form* (1.13), *where $u(\bar{H} - e^{-x})$ is a concave function.*

In contrast to the class S^+, the class S^- contains df's $H(x)$, which are discontinuous at the point $\bar{H}$. So, for example, the sf

$$u(x) = \begin{cases} x & \text{if } x < 0, \\ \infty & \text{if } x > 0, \end{cases}$$

satisfies the conditions, which are required in Theorem 4.4 and, therefore, the df

$$H(x) = \begin{cases} \Phi(x) & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

belongs to the class S^-.

Similarly to Theorem 4.2, the following proposition is valid:

THEOREM 4.6. *A df H, which is continuous at $\bar{H}$, belongs to S^- if and only if $H(-e^{-x})$ belongs to S^*.*

5. Auxiliary propositions

LEMMA 5.1. *Let relations* (1.6) *and* (2.2) *hold, where u is a proper sf. Then*:

(1) *The partial limits of the sequence* $\{b_n/a_n\}$ *must be outside the open interval* $(-\bar{u}, -\underline{u})$.

(2) *If in addition,* $\underline{u} < 0$, *then for each subsequence* n' *such that* $b'_n \to b$ $(|b| < \infty)$ *necessarily* $a_{n'} \to 0$.

PROOF. (1) Let us assume, for instance, that for some subsequence n'

$$b_{n'}/a_{n'} \to \beta, \quad -\bar{u} < \beta < -\underline{u}.$$

Then

$$u_{n'}(a_{n'}x + \beta a_{n'}) \to u(x) \tag{5.1}$$

and for every i and $x > \underline{u}$

$$F_i(a_{n'}x + \beta a_{n'}) \to 1.$$

In particular, since $\underline{u} < -\beta < \bar{u}$, then for every i

$$F_i(0) = 1. \tag{5.2}$$

Let ξ $(-\beta < \xi < \bar{u})$ be any continuity point of $u(x)$. Then by (5.1)

$$u_{n'}(a_{n'}\xi + a_{n'}\beta) \to u(\xi).$$

On the other hand, since $\xi + \beta > 0$, by (5.2) for every i

$$F_i(a_{n'}\xi + a_{n'}\beta) = 1$$

and therefore

$$u_{n'}(a_{n'}\xi + a_{n'}\beta) = \sqrt{p(n')} \to \infty,$$

which is impossible, since $\xi < \bar{u}$.

(2) Assuming $a_{n'} \to a$, $0 < a < \infty$, we obtain

$$u_{n'}(ax + b) \to u(x)$$

and for every i and $x > \underline{u}$, $F_i(ax + b) = 1$. Therefore

$$u_{n'}(ax + b) = \sqrt{p(n')} \to \infty$$

for any $x > \underline{u}$, which is impossible since u is a proper sf.

If we assume $a_{n'} \to \infty$, then for every x $(\underline{u} < x < 0)$ we shall have $F_i(a_{n'}x + b_{n'}) \to 0$, which contradicts (1.6).

LEMMA 5.2. *Let relations* (1.6), (1.10) *and* (2.2) *hold, where u is a proper sf and let*

$$\underline{u} < 0 \leqq \bar{u}. \tag{5.3}$$

Then there exists a non-decreasing sequence $\{\beta_n\}$ *such that also*

$$u_n(a_n x + \beta_n) \to u(x). \tag{5.4}$$

PROOF. First we shall prove that for every integer-valued function $m = m(n) \geqq 1$

$$\liminf (b_{n+m} - b_n)/a_n \geqq 0. \tag{5.5}$$

The assumptions of our Lemma hold for any subsequence. Therefore, in order to simplify the notation, it is enough to prove the impossibility of

$$(b_{n+m} - b_n)/a_n \to \beta, \qquad a_{n+m}/a_n \to \alpha,$$

where $-\infty \leqq \beta < 0$, $0 \leqq \alpha \leqq \infty$.

Assuming $\beta = -\infty$, for any $M > 0$

$$b_{n+m} < -Ma_n + b_n.$$

Hence, by the monotonicity of $u_n(x)$ and condition (1.10) (see Remark 2.6),

$$u_{n+m}(b_{n+m}) \leqq u_{n+m}(-Ma_n + b_n) = u_n(-Ma_n + b_n).$$

By $n \to \infty$ we obtain $u(0-) \leqq u(-\infty) = -\infty$, with contradicts assumption (5.3).

Let $-\infty < \beta < 0$. Due to (1.10)

$$u_{n+m}(a_{n+m}x + b_{n+m}) \leqq u_n\left(a_n\left(\frac{a_{n+m}}{a_n}x + \frac{b_{n+m} - b_n}{a_n}\right) + b_n\right). \tag{5.6}$$

For $\alpha = 0$ we conclude that $u(x) \leqq u(\beta +)$ for every x, which again contradicts (5.3), since $\beta < 0$.

For $\alpha = \infty$ obtain from (5.6) $u(x) \leqq u(-\infty) = -\infty$ for each $x < 0$. Hence $\underline{u} \geqq 0$, which is impossible.

Finally, the assumption $0 < \alpha < \infty$ implies

$$u(x) \leqq u(\alpha x + \beta). \tag{5.7}$$

If $0 < \alpha < 1$, then by iteration of the above inequality we have for every n

$$u(x) \leqq u(\alpha^n x + \beta(1 + \alpha + \cdots + \alpha^{n-1})).$$

Hence, as $n \to \infty$ we obtain for every x

$$u(x) \leqq u(\beta/(1-\alpha)),$$

which is impossible since $\beta/(1-\alpha)<0$ and $\bar{u} \geqq 0$.

Similarly, if $\alpha > 1$, then we rewrite (5.7) as

$$u\left(\frac{x-\beta}{\alpha}\right) \leqq u(x)$$

and by iteration we obtain for every x

$$u(x) \geqq u(-\beta/(1-\alpha)),$$

which is again impossible since $-\beta/(1-\alpha)>0$ and $\underline{u}<0$.

Obviously, $\alpha = 1$ is impossible since $u(x)$ is an sf.

Thus we proved (5.5), which implies that $\liminf b_n > -\infty$.

Now, let us define a sequence $\{\beta_n\}$ by

$$\beta_n = \inf(b_n, b_{n+1}, \dots), \qquad n = 1, 2, \dots .$$

Obviously $\beta_n \leqq b_n$ and $\beta_n \leqq \beta_{n+1}$.

In order to prove (5.4) we have to show that

$$(b_n - \beta_n)/a_n \to 0.$$

Indeed, if for some subsequence n' and some $\varepsilon > 0$ we have

$$(b_{n'} - \beta_{n'})/a_{n'} > \varepsilon,$$

then for every n' there exists among the terms of the reduced sequence $\{b_{n'+k}\}$ $(k = 1, 2, \dots)$ a term $b_{n'+m'}$ such that $(b_{n'} - b_{n'+m'})/a_{n'} > \varepsilon > 0$, which contradicts (5.5).

Corollary 5.1. *Under the conditions of Lemma 5.2 there exists a sequence of df's $\{\bar{F}_n(x)\}$ and a numerical sequence $\{\bar{b}_n\}$ such that*

$$\bar{u}_n(a_n x + b_n) = \left[\sum_{i=1}^{n} \bar{F}_i(a_n x + \bar{b}_n) - n\right] \Big/ \sqrt{p(n)} + \sqrt{p(n)} \to u(x),$$

$$\text{(5.8)} \qquad \bar{b}_n \leqq \bar{b}_{n+1},$$

and either

$$\text{(5.9)} \qquad \bar{b}_n \to \infty, \qquad \liminf(\bar{b}_n/a_n) \geqq -\underline{u}$$

or

$$\text{(5.10)} \qquad \bar{b}_n \to 0, \quad a_n \to 0, \qquad \limsup(\bar{b}_n/a_n) \leqq -\bar{u}.$$

PROOF. It follows from Lemma 5.2 that the initial $\{b_n\}$ may be replaced by a non-decreasing sequence $\{\beta_n\}$. Let $\beta_n \to b$, then $b > -\infty$.

If $b = \infty$, we put

$$\bar{b}_n = \beta_n, \qquad \tilde{F}_n(x) = F_n(x)$$

and the inequality (5.9) follows from the condition (5.3) and Lemma 5.1 (1).

If $|b| < \infty$, then we put

$$\bar{b}_n = \beta_n - b, \qquad \tilde{F}_n(x) = F_n(x+b)$$

and we obtain (5.10) by Lemma 5.1 (2).

6. The proof of Theorem 1.1

Taking into account Theorems 3.1 and 4.5 and Corollary 4.1, it is enough to show that the class S is the union of S^* and $\hat{S}^-$.

Let H be a df in S. Without loss of generality we may assume that

$$\underline{H} < 0 < \bar{H}. \tag{6.1}$$

By Theorem 2.1 H must be of the form (1.13), where u is a proper sf, which can be a limit in (2.2), where u_n is given by (1.8). By (6.1) the sf u satisfies condition (5.3). Due to Definition 1.1 and Corollary 5.1 we may assume that the numerical sequences $\{a_n, b_n\}$ satisfy condition (5.8) and either (5.9) or (5.10).

1. Let (5.8) and (5.9) hold. We shall prove that in this case H must belong to S^*.

Let $\beta > 0$ be any number and let us consider the sets of integers k

$$E_n(\beta) = \{k : (b_{n+k-1} - b_n)/a_n \leqq \beta\}, \tag{6.2}$$

which are obviously non-empty and bounded, since $b_n \to \infty$.

For the given β we define an integer-valued function $m = m_\beta(n)$ by

$$m = \max E_n(\beta). \tag{6.3}$$

Thus, from some n on we have

$$\beta < (b_{n+m} - b_n)/a_n \leqq \beta + \frac{b_{n+m} - b_{n+m-1}}{a_{n+m}} \cdot \frac{a_{n+m}}{a_n}, \tag{6.4}$$

i.e.,

$$\beta a_n / a_{n+m} < (b_{n+m} - b_n)/a_{n+m} \leqq \beta a_n / a_{n+m} + (b_{n+m} - b_{n+m-1})/a_{n+m}. \tag{6.5}$$

Our assumptions imply

$$\limsup a_{n+m}/a_n < \infty. \tag{6.6}$$

Indeed, if for some subsequence n', $m' = m(n')$,

$$a_{n'+m'}/a_{n'} \to \infty, \tag{6.7}$$

then by (6.5) $(b_{n'+m'} - b_{n'})/a_{n'+m'} \to 0$ and consequently

$$u_{n'+m'}(a_{n'+m'}x + b_{n'}) \to u(x). \tag{6.8}$$

On the other hand, by (6.7), for any continuity points of u $M < \varepsilon < 0$, we have from some b' on $a_{n'+m'}\ \varepsilon < a_{n'}M$. Hence, by (1.10)

$$u_{n'+m'}(a_{n'+m'}\varepsilon + b_{n'}) \leqq u_{n'}(a_{n'}M + b_{n'})$$

and consequently, in view of (6.8), $u(\varepsilon) \leqq u(M)$.

By $\varepsilon \to 0$, $M \to -\infty$ we obtain $u(0-) = -\infty$, which contradicts condition (5.3). Thus we have proved (6.6), and by (2.4) and (6.4) we get that

$$(b_{n+m} - b_n)/a_n \to \beta. \tag{6.9}$$

LEMMA 6.1. *Let $\{a_n > 0, b_n\}$ be numerical sequences. If $b_n \to \infty$ and for some integer-valued function $m = m(n) \geqq 1$ we have* (6.9), *where $\beta > 0$, then*

$$\limsup a_{n+m}/a_n \geqq 1. \tag{6.10}$$

PROOF. Let us assume that from some N_1 on

$$a_{n+m}/a_n < q < 1. \tag{6.11}$$

Let $b > \beta$. Then by (6.9) from some N_2 on

$$(b_{n+m} - b_n)/a_n < b. \tag{6.12}$$

Let $n_0 = \max(N_1, N_2)$, $n_s = n_{s-1} + m(n_{s-1})$, $s = 1, 2, \ldots$. Since $m(n) \geqq 1$, $\{n_s\}$ is an infinite sequence of integers such that

$$b_{n_s} - b_{n_{s-1}} < b a_{n_{s-1}}, \qquad a_{n_s}/a_{n_{s-1}} < q < 1.$$

Hence, for any k

$$b_{n_k} - b_{n_0} = \sum_{s=1}^{k} (b_{n_s} - b_{n_{s-1}}) < b \sum_{s=1}^{k} a_{n_{s-1}} < b a_{n_0} \sum_{s=1}^{k} q^{s-1} < b a_{n_0}/(1-q), \tag{6.13}$$

which contradicts our assumption that $b_n \to \infty$. Thus we have proved (6.10).

The expression of $u_{n+m}(a_{n+m}x+b_{n+m})$ can be rewritten in the form

(6.14) $$u_{n+m}(a_{n+m}x+b_{n+m})=[p(n)/p(n+m)]^{1/2}u_n(a_{n+m}x+b_{n+m})+\varphi_n(x),$$

where

$$\varphi_n(x)=\left[\sum_{i=n+1}^{n+m}F_i(a_{n+m}x+b_{n+m})+p(n+m)-p(n)-m\right]\Big/\sqrt{p(n+m)}$$

is a non-decreasing function. By (6.6) and (6.10) there exists a subsequence n' such that

$$a_{n'+m'}/a_{n'}\to\alpha,$$

where $1\leqq\alpha<\infty$. Passing to the limit in (6.14) over n' and taking into account (6.9) and (1.12) we conclude that the sf u must satisfy the following condition: for every $\beta>0$ there exists an $\alpha=\alpha(\beta)\geqq 1$ such that the difference

(6.15) $$\varphi_\beta(x)=u(x)-u(\alpha x+\beta)$$

is a non-decreasing function. Thus it remains to prove that we may take $\alpha=\alpha(\beta)=1$.

LEMMA 6.2. *If the sf u satisfies condition* (6.15), *then for any $\xi>\underline{u}$ this condition is also satisfied by $\tilde u(x)=u(x+\xi)$.*

PROOF. Let us denote $\bar b_n=b_n+a_n\xi$. Then

$$u_n(a_nx+\bar b_n)\to\tilde u(x).$$

It follows from our previous argument that it is enough to show that $(\bar b_{n+1}-\bar b_n)/a_n\to 0$ and $\bar b_n\to\infty$. The first relation follows instantly from (2.4). For the second relation it is enough to consider the case when the sequence $\{a_n\}$ is unbounded and $\underline{u}<\xi<0$. But by (5.9)

$$\liminf \bar b_n/a_n\geqq-\underline{u}+\xi>0,$$

which proves that $\bar b_n\to\infty$ also in the case when $\{a_n\}$ is unbounded. Thus the Lemma is proved.

Now, let us return to the relation (6.15). If $\alpha\geqq 1$, $x_1<0<y_1$, then $\alpha x_1+\beta\leqq x_1+\beta$, $\alpha y_1+\beta\geqq y_1+\beta$. Therefore

$$u(x_1)-u(x_1+\beta)\leqq u(x_1)-u(\alpha x_1+\beta)\leqq u(y_1)-u(\alpha y_1+\beta)\leqq u(y_1)-u(y_1+\beta),$$

i.e.

$$u(x_1) - u(x_1 + \beta) \leqq u(y_1) - u(y_1 + \beta). \tag{6.16}$$

If $x_1 < y_1 < 0$ or $0 < x_1 < y_1$, then we consider the sf $\tilde{u}(x) = u(x + \xi)$, where ξ, $x_1 < \xi < y_1$, is any number. By Lemma 6.2, also the sf $\tilde{u}(x)$ has property (6.15), with $\tilde{\alpha} = \tilde{\alpha}(\beta) \geqq 1$, and we may take $\tilde{\alpha} = \tilde{\alpha}(\beta) = 1$, if $x < 0 < y$. Let us take

$$x = x_1 - \xi, \qquad y = y_1 - \xi,$$

then $x < 0 < y$ and consequently

$$\tilde{u}(x) - \tilde{u}(x + \beta) \leqq \tilde{u}(y) - \tilde{u}(y + \beta),$$

i.e., (6.16).

Thus we have proved that for any $x \leqq y$ we can put $\alpha = 1$ in (6.15). Hence, by Theorem 3.1, $H = \Phi(u)$ must belong to S^*.

2. Let (5.8) and (5.10) hold, where $\bar{u} = \infty$, i.e.,

$$b_n / a_n \to -\infty. \tag{6.17}$$

We shall prove that in this case $H = \Phi(u)$ belongs again to S^*.

We consider again the sets (6.2). By (6.17), for any $\beta > 0$ we have from some N on $-b_n/a_n > \beta$. Hence, for $n > N$ the sets $E_n(\beta)$ are bounded. We define the integer-valued function $m = m_\beta(n)$ by (6.3) (for $n \leqq N$ we can put $m(n) = 1$). Thus using the same argument as in the previous case, we obtain relation (6.9). Now we need the following

LEMMA 6.3. *Let $\{a_n > 0, b_n\}$ be numerical sequences. If $b_n \to 0$ and (6.17) hold, and for some $m = m(n) \geqq 1$ we have (6.9), then also (6.10).*

PROOF. Let us assume that from some N_1 on we have (6.11) and (6.12). By (6.17) we have from some N_2 on

$$b_n / a_n < -\frac{2b}{1-q}\,. \tag{6.18}$$

Now, considering the subsequence $\{n_s\}$, defined in the proof of Lemma 6.1, we obtain the inequality (6.13). Hence, by (6.18)

$$b_{n_k}/a_{n_0} < b_{n_0}/a_{n_0} + b/(1-q) < -b/(1-q) < 0,$$

which is impossible, since $b_{n_k} \to 0$. Thus we proved (6.10).

Using the above Lemma, we conclude the necessity of (6.15). Under the conditions of this case, for any ξ the sf $\tilde{u}(x) = u(x + \xi)$ also has property (6.15), which by the argument of the previous case enables us to take $\alpha = 1$ in (6.15).

3. Finally, let (5.8) and (5.10) hold, where $\bar{u} < \infty$. Let

$$-\infty < b = \limsup b_n / a_n < -\bar{u}.$$

We shall show that in the present case $H = \Phi(u)$ belongs to $\hat{S}^-$.

First we will prove that u has the following property: for every α $(0 < \alpha < 1)$, there exists a $\beta = \beta(\alpha) \geqq 0$ such that the difference

$$\varphi_\alpha(x) = u(x) - u(\alpha x + \beta) \tag{6.19}$$

is a non-decreasing function and

$$\bar{u} \leqq \beta/(1-\alpha) \leqq -b. \tag{6.20}$$

Let $0 < \alpha < 1$ be any number. Since in the present case $a_n \to 0$, by (2.4) there exists an integer-valued function $m = m_\alpha(n)$ such that

$$a_{n+m} / a_n \to \alpha. \tag{6.21}$$

Let n' be such a subsequence that

$$b_{n'} / a_{n'} \to b.$$

Then by (6.21)

$$0 \leqq \liminf (b_{n'+m'} - b_{n'})/a_{n'} \leqq \limsup (b_{n'+m'} - b_{n'})/a_{n'} \leqq -b(1-\alpha).$$

Hence we may assume that

$$(b_{n'+m'} - b_{n'})/a_{n'} \to \beta, \qquad 0 \leqq \beta \leqq -b(1-\alpha). \tag{6.22}$$

Using the representation (6.14), on account of (6.21), (6.22) and (1.12) we obtain relation (6.19), where by (1.10) we have (5.7). By iterating n times the substitution of $\alpha x + \beta$ for x in (5.7), as $n \to \infty$ we conclude that $u(x) \leqq u(\beta/(1-\alpha))$ for any x. Thus $\bar{u} \leqq \beta/(1-\alpha)$ and by (6.22) we obtain (6.20).

Now, let $\{\alpha_n\}$ be any sequence such that

$$\alpha_n \to 1, \qquad 0 < \alpha_n < 1 \tag{6.23}$$

and let $\beta_n = \beta(\alpha_n)$ be the corresponding sequence such that

$$\varphi_n(x) = u(x) - u(\alpha_n x + \beta_n) \tag{6.24}$$

is a non-decreasing function. By (6.20) we may assume

$$\beta_n \to 0, \qquad \beta_n/(1-\alpha_n) \to -b_0, \tag{6.25}$$

where $\bar{u} \leqq -b_0 \leqq -b$.

The sequences $\{\alpha_n, \beta_n\}$ given by (6.23) and (6.25) should be considered fixed in the sequel. Let us put in (6.24) $x + \beta_n/(1-\alpha_n)$ instead of x. Then we obtain

$$u(x+\beta_n/(1-\alpha_n)) - u(\alpha_n x + \beta_n/(1-\alpha_n)) = \varphi_n(x+\beta_n/(1-\alpha_n)).$$

Hence, repeating $(s-1)$ times the substitution of $\alpha_n x$ for x and then summing, we obtain

$$u(x+\beta_n/(1-\alpha_n)) - u(\alpha_n^s x + \beta_n/(1-\alpha_n)) = \varphi_{n,s}(x), \tag{6.26}$$

where

$$\varphi_{n,s}(x) = \sum_{k=0}^{s-1} \varphi_n(\alpha_n^k x + \beta_n/(1-\alpha_n)) \tag{6.27}$$

is a non-decreasing function for any integers n and s.

Now, for a given α $(0<\alpha<1)$ let us take in (6.26) the integer-valued function

$$s = s(n) = s_\alpha(n) = [\log\alpha/\log\alpha_n].$$

We have $\alpha_n^{s(n)} \to \alpha$. Therefore, passing to the limit in (6.26) by $n\to\infty$ we obtain

$$u(x-b_0) - u(\alpha x - b_0) = \varphi_\alpha(x),$$

where $\varphi_\alpha(x)$ is the limit of the expression (6.27). Obviously, $\varphi_\alpha(x)$ is a non-decreasing function.

Thus we have proved that the df $\Phi(u(x-b_0))$ belongs to S^-. Hence, $\bar{H} = -b_0$ and $h = \Phi(u)$ belongs to $\hat{S}^-$.

This completes the proof.

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